MATHEMATICS

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UNIQUENESS THEOREM OF SOLUTION THE INTEGRAL GEOMETRY PROBLEM FOR FAMILY CURVES IN MULTIDIMENSIONAL SPACE

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Abstract

In this article the following class of integral geometry problems is considered: about the function reconstruction, shared by the integrals on some set of curves. These problems are correlated with several applications. In order to study internal earth structure multiple explosions are held on Earth surface. Then, fluctuations regimes of earth surface are measured on equipment for each explosion. The purpose of research is to determine distribution of physical parameters inside the Earth according to equipment measurements correlated with laws on dissemination of seismic waves. The most clear functional of such equipment is arrival time of seismic wave, which exactly serves as a base for interpretation practice. It is known that linearized problem of seismic-exploration data interpretation is actually the problem of integral geometry. Integral geometry also includes problems related to radiography, particularly interpretation problem of X-ray images. For instance, an X-ray film darkening is functionally correlated with absorption coefficient is also actually an integral geometry problem. In this case, it is required to determine the function if the integrals of this function on set of rays were set. An integral geometry problem in multidimensional space is studied in this work. The theorem of solution uniqueness is proven for the considered integral geometry problem.

Keywords: integral geometry, family of curves, integral equation, solution, uniqueness.

Let's consider the following problem of integral geometry [1, 2]

$$v(\xi,\eta,\zeta) = \iint\limits_{S(\xi,\eta,\zeta)} u(x,y,z)\,dS, \qquad (1)$$

where $S(\xi,\eta,\zeta)$ – family of cones $(\zeta - z)^2 = (\xi - x)^2 + (\eta - y)^2 (0 \le z \le \zeta)$ or $z = \zeta - \sqrt{(\xi - x)^2 + (\eta - y)^2}$ with apexes in points (ξ,η,ζ), basing on $z = 0$ plane.

Considering that

$$p = z'_x = \frac{\xi - x}{\sqrt{(\xi - x)^2 + (\eta - y)^2}}, \quad q = z'_y = \frac{\eta - y}{\sqrt{(\xi - x)^2 + (\eta - y)^2}}.$$

$$\sqrt{1 + p^2 + q^2} = \sqrt{2}$$

surface integral (1) may be traced to repeated integral

$$v(\xi,\eta,\zeta) = \iint\limits_{D(\xi,\eta,\zeta)} u\left(x, y, \zeta - \sqrt{(\xi - x)^2 + (\eta - y)^2}\right) dxdy.$$

If we introduce polar coordinate system $\xi = x + r\cos\varphi$, $\eta = y + r\sin\varphi$, then we have

$$v(\xi,\eta,\zeta) = \sqrt{2}\int_0^{2\pi}\int_0^{\zeta} u(\xi - r\cos\varphi, \eta - r\sin\varphi, \zeta - r)\,r\,dr\,d\varphi.$$

We apply Fourier transform to both members of equation on variables ξ, η:

$$V(\lambda,\mu,\zeta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v(\xi,\eta,\zeta)\,e^{i(\lambda\xi + \mu\eta)}\,d\xi d\eta =$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^{\zeta} r\,dr\,d\varphi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(\xi - r\cos\varphi, \eta - r\sin\varphi, \zeta - r)\,e^{i(\lambda\xi + \mu\eta)}\,d\xi d\eta.$$

Further, introducing change of variables $\xi - r\cos\varphi = t$, $\eta - r\sin\varphi = \tau$ we modify last-mentioned equation into

$$V(\lambda,\mu,\zeta) = \sqrt{2}\int_0^{2\pi}\int_0^{\zeta} r\,e^{ir(\lambda\cos\varphi + \mu\sin\varphi)}\,U(\lambda,\mu,\zeta - r)\,dr\,d\varphi.$$

where $U(\lambda,\mu,z)$ – Fourier transform of function $u(x,y,z)$ on variables x, y. Changing order of integration we obtain Volterra integral equation of the first kind concerning the function $U(\lambda,\mu,z)$:

$$v(\lambda,\mu,\zeta) = \int_0^{\zeta} rK(\lambda,\mu,r)\,u(\lambda,\mu,\zeta - r)\,dr, \qquad (2)$$

where

$$K(\lambda,\mu,r) = \sqrt{2}\int_0^{2\pi} e^{ir(\lambda\cos\varphi + \mu\sin\varphi)}\,d\varphi.$$

Change of $\zeta - r = \rho$ allows obtain equation

$$v(\lambda,\mu,\zeta) = \int_0^{\zeta} (\zeta - r)K(\lambda,\mu,\zeta-\rho)\,u(\lambda,\mu,\rho)\,d\rho.$$

Differentiating this equation on ζ we obtain

$$v'_{\zeta}(\lambda,\mu,\zeta) = \int_0^{\zeta} \left[K(\lambda,\mu,\zeta-\rho) + (\zeta - r)K'_{\zeta}(\lambda,\mu,\zeta-\rho)\right]u(\lambda,\mu,\rho)\,d\rho.$$

Having differentiated one more time on ζ we arrive at equation

$$v''_{\zeta\zeta}(\lambda,\mu,\zeta) = K(\lambda,\mu,0)u(\lambda,\mu,\zeta) + \int_0^{\zeta}\left[2K'_{\zeta}(\lambda,\mu,\zeta-\rho) + (\zeta - r)K''_{\zeta\zeta}(\lambda,\mu,\zeta-\rho)\right]u(\lambda,\mu,\rho)\,d\rho. \qquad (3)$$

Let us calculate integral $K(\lambda,\mu,r) = 2\sqrt{2}\pi[J_0(\lambda r) + J_0(\mu r)]$ [3, formula (3.715)] where $J_0(x)$ – zero-order Bessel function of the first kind. As is known, $J_0(0) = 1$, therefore $K(\lambda,\mu,0) = 4\sqrt{2}\pi$. Thus, equation (3) may be recorded in the form of Volterra integral equation of the second kind [4-6]

$$v''_{\zeta\zeta}(\lambda,\mu,\zeta) = 4\sqrt{2}\pi u(\lambda,\mu,\zeta) + \int_0^{\zeta}\Psi(\lambda,\mu,\zeta-\rho)\,u(\lambda,\mu,\rho)\,d\rho,$$

$$\Psi(\lambda,\mu,\zeta-\rho) = 4\sqrt{2}\pi[\lambda J'_0(\lambda(\zeta-\rho)) + \mu J'_0(\mu(\zeta-\rho))] + 2\sqrt{2}\pi(\zeta-\rho)[\lambda^2 J''_0(\lambda(\zeta-\rho)) + \mu^2 J''_0(\mu(\zeta-\rho))].$$

Thus,

Theorem 1 has been proven. If function $v(\xi,\eta,\zeta)$ has finite continuity on variables ξ,η and twice differentiated on ζ, then solution $u(x,y,z)$ of considered problem of integral geometry is unique in the class of finite continous functions.

Let us consider more common problem of integral geometry

$$v(\xi,\eta) = \int\limits_{S(\xi,\eta)} u(x,y)\,dS. \qquad (4)$$

where $\xi = (\xi_1,\xi_2,\dots,\xi_n)$, $x = (x_1,x_2,\dots,x_n)$, $S(\xi,\eta)$ – family of surfaces

$$|\eta - y| = |x - \xi|\,(0 \le y \le \eta)\ \text{или}\ y = \eta - \sqrt{\sum_{i=1}^{n}(x_i-\xi_i)^2}.$$

Considering that

$$p_i = y'_{x_i} = -(x_i-\xi_i)\Big/\sqrt{\sum_{i=1}^{n}(x_i-\xi_i)^2}\ (i=\overline{1,n}),\quad \sqrt{1+\sum_{i=1}^{n}p_i^2} = \sqrt{2}$$

we modify surface integral (4) into

$$v(\xi,\eta) = \sqrt{2}\int\limits_{D(\xi,\eta)} u(x,\eta - |x-\xi|)\,dx,$$

where $D(\xi,\eta)$ – projection of a surface $S(\xi,\eta)$ to hyperplane $y = 0$.

Let us introduce variables $x_i = \xi_i + r\cos\varphi_i$ $(i=\overline{1,n})$, where $\cos\varphi_i$ $(i=\overline{1,n})$ – directional cosines of normal vector $\vec{\psi}$ to specified surface of $S(\xi,\eta)$; $r = |\vec{\psi}|$ family. Taking into account proportion

$$\sum_{i=1}^{n}\cos^2\varphi_i = 1$$

we obtain

$$x_i = \xi_i + r\cos\varphi_i\,(i=\overline{1,n-1}),\qquad x_n = \xi_n + r\sqrt{1-\sum_{i=1}^{n-1}\cos^2\varphi_i}.$$

Jacobian of such transformation (appendix 1) $R(r,\varphi) = r^{n-1}S(\varphi)$, where

$$\varphi = (\varphi_1,\varphi_2,\dots,\varphi_{n-1}),\quad S(\varphi) = \prod_{i=1}^{n-1}\sin\varphi_i\Big/\sqrt{1-\sum_{i=1}^{n-1}\cos^2\varphi_i}.$$

Then

$$v(\xi,\eta) = \sqrt{2}\int_0^{2\pi}\int_0^{\eta} u(\xi - r\psi, \eta - r)R(r,\varphi)\,dr\,d\varphi.$$

We apply Fourier transformationto both members of equation on vector ξ:

$$v(\lambda,\eta) = \int_{-\infty}^{+\infty} e^{i(\lambda,\xi)}d\xi\int_0^{2\pi}\int_0^{\eta}\sqrt{2}\,u(\xi - r\psi, \eta - r)R(r,\varphi)\,dr\,d\varphi.$$

Now, we change order of integration

$$v(\lambda,\eta) = \sqrt{2}\int_0^{2\pi}\int_0^{\eta} R(r,\varphi)\,dr\,d\varphi\int_{-\infty}^{+\infty} u(\xi - r\psi, \eta - r)e^{i(\lambda,\xi)}d\xi.$$

By changing $\xi - r\psi = t$ $(t = (t_1, t_2, \ldots, t_n))$ we have

$$v(\lambda,\eta) = \sqrt{2}\int_0^{2\pi}\int_0^{\eta} R(r,\varphi)e^{i(\lambda,r\psi)}\,dr\,d\varphi\int_{-\infty}^{+\infty} u(t, \eta - r)e^{i(\lambda,t)}dt,$$

it follows from here that

$$v(\lambda,\eta) = \int_0^{\eta} r^{n-1}\left(\sqrt{2}\int_0^{2\pi} S(\varphi)e^{ir(\lambda,\psi)}d\varphi\right)\tilde{u}(\lambda, \eta - r)\,dr.$$

where $\tilde{u}$ – Fourier transformation of function u on vector ξ.

Change of variable $\eta - r = \rho$ allows record the last-mentioned equation into

$$v(\lambda,\eta) = \int_0^{\eta}(\eta - \rho)^{n-1}T(\lambda, \eta - \rho)\tilde{u}(\lambda,\rho)\,d\rho.$$

or

$$v(\lambda,\eta) = \int_0^{\eta} K(\lambda, \eta - \rho)\tilde{u}(\lambda,\rho)\,d\rho, \qquad (5)$$

where

$$K(\lambda, \eta - \rho) = (\eta - \rho)^{n-1}T(\lambda, \eta - \rho), \; T(\lambda, \eta - \rho) = \sqrt{2}\int_0^{2\pi} S(\varphi)e^{i(\eta-\rho)(\lambda,\psi)}d\varphi.$$

We differentiate family of Volterra integral equations of the first kind on η

$$v'_\eta(\lambda,\eta) = K(\lambda,0)\tilde{u}(\lambda,\eta) + \int_0^{\eta} K_\eta^{(1)}(\lambda, \eta - \rho)\tilde{u}(\lambda,\rho)\,d\rho.$$

Taking into account that $K(\lambda,0)=0$, let us differentiate the last-mentioned equation one more time on η

$$u''_{\eta\eta}(\lambda,\eta)=K_\eta^{(1)}(\lambda,0)u(\lambda,\rho)+\int_0^\eta K_\eta^{(2)}(\lambda,\eta-\rho)u(\lambda,\rho)d\rho.$$

From formulas

$$K_\eta^{(j)}(\lambda,\eta-\rho)=\frac{(n-1)!}{(n-j-1)!}(\eta-\rho)^{n-j-1}T(\lambda,\eta-\rho)+$$
$$+C_j^1\frac{(n-1)!}{(n-j)!}(\eta-\rho)^{n-j}T_\eta^{(1)}(\lambda,\eta-\rho)+$$
$$+C_j^2\frac{(n-1)!}{(n-j+1)!}(\eta-\rho)^{n-j+1}T_\eta^{(2)}(\lambda,\eta-\rho)+\cdots+$$
$$+C_j^{j-1}(n-1)(\eta-\rho)^{n-2}T_\eta^{(j-1)}(\lambda,\eta-\rho)+(\eta-\rho)^{n-1}T_\eta^{(j)}(\lambda,\eta-\rho),$$

where C_j^k – number of combinations,

$$T_\eta^{(j)}(\lambda,\eta-\rho)=i^j\sqrt{2}\int_0^{2\pi}S(\varphi)e^{i(\eta-\rho)(\lambda,\varphi)}(\lambda,\psi)^j d\varphi,$$

it follows that

$$K_\eta^{(1)}(\lambda,0)=K_\eta^{(2)}(\lambda,0)=\cdots=K_\eta^{(n-2)}(\lambda,0)=0.$$

From formula

$$K_\eta^{(n-1)}(\lambda,\eta-\rho)=(n-1)!\,T(\lambda,\eta-\rho)+$$
$$+C_{n-1}^1(n-1)!(\eta-\rho)T_\eta^{(1)}(\lambda,\eta-\rho)+\cdots+(\eta-\rho)^{n-1}T_\eta^{(n-1)}(\lambda,\eta-\rho)$$

we obtain

$$K_\eta^{(n-1)}(\lambda,0)=(n-1)!\,T(\lambda,0)=(n-1)!\sqrt{2}\int_0^{2\pi}S(\varphi)d\varphi\neq 0,$$

as we can prove in equation (appendix 2)

$$\int_0^{2\pi}S(\varphi)\,d\varphi\geq(2\pi)^{n-1}.$$

Thus, by differentiating integral equation (5) totally n times on η we obtain Volterra integral equation of the second kind

$$u_\eta^{(n)}(\lambda,\eta)=K_\eta^{(n-1)}(\lambda,0)u(\lambda,\rho)+\int_0^\eta K_\eta^{(n)}(\lambda,\eta-\rho)u(\lambda,\rho)d\rho.$$

or

$$\frac{v_\eta^{(n)}(\lambda,\eta)}{K_\eta^{(n-1)}(\lambda,0)} = u(\lambda,\rho) + \int_0^\eta \frac{K_\eta^{(n)}(\lambda,\eta-\rho)}{K_\eta^{(n-1)}(\lambda,0)} u(\lambda,\rho)d\rho.$$

Therefore,

Theorem 2 is true. If $v(\xi,\eta)$ has a finite continuity through vector ξ and is n times differentiable through η, then solution $u(x,y)$ of problem (4) is uniquein the class of finite continous functions.

Appendix 1.

Jacobian

$$R(r,\varphi) = \begin{vmatrix} x'_{1_r} & x'_{1_{\varphi_1}} & x'_{1_{\varphi_2}} & \dots & x'_{1_{\varphi_{n-1}}} \\ x'_{2_r} & x'_{2_{\varphi_1}} & x'_{2_{\varphi_2}} & \dots & x'_{2_{\varphi_{n-1}}} \\ \dots & \dots & \dots & \dots & \dots \\ x'_{n-1_r} & x'_{n-1_{\varphi_1}} & x'_{n-1_{\varphi_2}} & \dots & x'_{n-1_{\varphi_{n-1}}} \\ x'_{n_r} & x'_{n_{\varphi_1}} & x'_{n_{\varphi_2}} & \dots & x'_{n_{\varphi_{n-1}}} \end{vmatrix} =$$

$$= \begin{vmatrix} cos\varphi_1 & rsin\varphi_1 & 0 & \dots & 0 \\ cos\varphi_2 & 0 & rsin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ cos\varphi_{n-1} & 0 & 0 & \dots & rsin\varphi_{n-1} \\ cos\varphi_n & \frac{rcos\varphi_1 sin\varphi_1}{cos\varphi_n} & \frac{rcos\varphi_2 sin\varphi_2}{cos\varphi_n} & \dots & \frac{rcos\varphi_{n-1} sin\varphi_{n-1}}{cos\varphi_n} \end{vmatrix} =$$

$$= cos\varphi_1 \begin{vmatrix} 0 & rsin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & rsin\varphi_{n-1} \\ \frac{rcos\varphi_1 sin\varphi_1}{cos\varphi_n} & \frac{rcos\varphi_2 sin\varphi_2}{cos\varphi_n} & \dots & \frac{rcos\varphi_{n-1} sin\varphi_{n-1}}{cos\varphi_n} \end{vmatrix} -$$

$$- cos\varphi_2 \begin{vmatrix} rsin\varphi_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & rsin\varphi_{n-1} \\ \frac{rcos\varphi_1 sin\varphi_1}{cos\varphi_n} & \frac{rcos\varphi_2 sin\varphi_2}{cos\varphi_n} & \dots & \frac{rcos\varphi_{n-1} sin\varphi_{n-1}}{cos\varphi_n} \end{vmatrix} + \dots +$$

$$+ (-1)^{n-2} cos\varphi_{n-1} \begin{vmatrix} rsin\varphi_1 & 0 & \dots & 0 \\ 0 & rsin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{rcos\varphi_1 sin\varphi_1}{cos\varphi_n} & \frac{rcos\varphi_2 sin\varphi_2}{cos\varphi_n} & \dots & \frac{rcos\varphi_{n-1} sin\varphi_{n-1}}{cos\varphi_n} \end{vmatrix} +$$

$$+ (-1)^{n-1} cos\varphi_n \begin{vmatrix} rsin\varphi_1 & 0 & \dots & 0 \\ 0 & rsin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & rsin\varphi_{n-1} \end{vmatrix}$$

$$(-1)^{n-2}\frac{r\cos^2\varphi_1 \sin\varphi_1}{\cos\varphi_n}\begin{vmatrix} r\sin\varphi_2 & 0 & \dots & 0 \\ 0 & r\sin\varphi_3 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r\sin\varphi_{n-1} \end{vmatrix} +$$

$$+(-1)^{n-2}\frac{r\cos^2\varphi_2 \sin\varphi_2}{\cos\varphi_n}\begin{vmatrix} r\sin\varphi_1 & 0 & \dots & 0 \\ 0 & r\sin\varphi_3 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r\sin\varphi_{n-1} \end{vmatrix} + \dots +$$

$$+(-1)^{n-2}\frac{r\cos^2\varphi_{n-1} \sin\varphi_{n-1}}{\cos\varphi_n}\begin{vmatrix} r\sin\varphi_1 & 0 & \dots & 0 \\ 0 & r\sin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r\sin\varphi_{n-2} \end{vmatrix} +$$

$$+(-1)^{n-1}\cos\varphi_n\begin{vmatrix} r\sin\varphi_1 & 0 & \dots & 0 \\ 0 & r\sin\varphi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r\sin\varphi_{n-1} \end{vmatrix}=$$

$$=\frac{r^{n-1}\sin\varphi_1 \sin\varphi_2 \dots \sin\varphi_{n-1}}{\cos\varphi_n}(\cos^2\varphi_1 + \cos^2\varphi_2 + \dots + \cos^2\varphi_{n-1}) +$$

$$+ r^{n-1}\cos\varphi_n \sin\varphi_1 \sin\varphi_2 \dots \sin\varphi_{n-1} =$$

$$=\frac{r^{n-1}\sin\varphi_1 \sin\varphi_2 \dots \sin\varphi_{n-1}}{\cos\varphi_n}\sum_{k=1}^{n}\cos^2\varphi_k = \frac{r^{n-1}}{\cos\varphi_n}\prod_{i=1}^{n-1}\sin\varphi_i.$$

Appendix 2.

When $n = 2$

$$\int_0^{2\pi}\frac{\sin\varphi_1}{\sqrt{1-\cos^2\varphi_1}}d\varphi_1 = 2\pi.$$

When $n = 3$

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\sin\varphi_1 \sin\varphi_2}{\sqrt{1-\cos^2\varphi_1-\cos^2\varphi_2}}d\varphi_1 d\varphi_2 \geq$$

$$\geq \int_0^{2\pi}\int_0^{2\pi}\frac{\sin\varphi_1 \sin\varphi_2}{\sqrt{1-\cos^2\varphi_1-\cos^2\varphi_2+\cos^2\varphi_1\cos^2\varphi_2}}d\varphi_1 d\varphi_2 =$$

$$= \int_0^{2\pi}\int_0^{2\pi}\frac{\sin\varphi_1 \sin\varphi_2}{\sqrt{(1-\cos^2\varphi_1)(1-\cos^2\varphi_2)}}d\varphi_1 d\varphi_2 = (2\pi)^2.$$

When $n = 4$

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{sin\varphi_1 sin\varphi_2 sin\varphi_3}{\sqrt{1 \quad cos^2\varphi_1 \quad cos^2\varphi_2 \quad cos^2\varphi_3}}d\varphi_1 d\varphi_2 d\varphi_3 \geq$$

$$\geq \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{sin\varphi_1 sin\varphi_2 sin\varphi_3 d\varphi_1 d\varphi_2 d\varphi_3}{\sqrt{1 \quad cos^2\varphi_1 \quad cos^2\varphi_2 \quad cos^2\varphi_3 + cos^2\varphi_1(cos^2\varphi_2 + cos^2\varphi_3) + I}}$$

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{sin\varphi_1 sin\varphi_2 sin\varphi_3}{\sqrt{(1 \quad cos^2\varphi_1)(1 \quad cos^2\varphi_2)(1 \quad cos^2\varphi_3)}}d\varphi_1 d\varphi_2 d\varphi_3 \quad (2\pi)^3.$$

as far as

$$cos^2\varphi_1(cos^2\varphi_2 + cos^2\varphi_3) + I \geq 0,$$

where

$I \quad cos^2\varphi_1 cos^2\varphi_2 cos^2\varphi_3.$

According to mathematical induction method we assume when $n \quad k$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-2} sin\varphi_{k-1}}{\sqrt{1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-2} \quad cos^2\varphi_{k-1}}}d\varphi_1 \dots d\varphi_{k-2} d\varphi_{k-1} \geq$$

$$\geq (2\pi)^{k-1}.$$

Let us prove when $n \quad k+1$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-1} sin\varphi_k d\varphi_1 \dots d\varphi_{k-1} d\varphi_k}{\sqrt{1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-1} \quad cos^2\varphi_k}} \geq$$

$$\geq \int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-1} sin\varphi_k d\varphi_1 \dots d\varphi_{k-1} d\varphi_k}{\sqrt{1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-1} \quad cos^2\varphi_k + U}}$$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-1} sin\varphi_k d\varphi_1 \dots d\varphi_{k-1} d\varphi_k}{\sqrt{1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-1} \quad cos^2\varphi_k + U cos^2\varphi_k}}$$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-1} sin\varphi_k d\varphi_1 \dots d\varphi_{k-1} d\varphi_k}{\sqrt{1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-1} \quad cos^2\varphi_k(1 \quad U)}}$$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\frac{sin\varphi_1 \dots sin\varphi_{k-1} sin\varphi_k d\varphi_1 \dots d\varphi_{k-1} d\varphi_k}{\sqrt{(1 \quad cos^2\varphi_1 \quad \dots \quad cos^2\varphi_{k-1})(1 \quad cos^2\varphi_k)}} \geq$$

$$\geq (2\pi)^{k-1}\int_0^{2\pi} d\varphi_k \quad (2\pi)^k.$$

where $U \quad cos^2\varphi_1 + \dots + cos^2\varphi_{k-1}$. There fore, for any natural $n \geq 2$ inequality holds true

$$\int_0^{2\pi} S(\varphi)\, d\varphi \geq (2\pi)^{n-1}.$$

References:

[1] Lavrentev M.M., Romanov V.G., Shishatskij S.P. Nekorrektnye zadachi matematicheskoj fiziki I analiza. – Moskva: Nauka, 1980. – 286 s.
[2] Kabanikhin S.I. Obratnyeine korrektnye zadachi. – Novosibirsk: Sibirskoe nauchnoe izdatelstvo, 2008. – 460 s.
[3] Gradshtein I.S., Ryzhik I.M. Tablitsy integralov, sum, ryadov I proizvedenij. – Moskva: Nauka, 1971. – 1108 s.
[4] Mikhlin S.G. Lektsii po lineinym integralnym uravneniyam. – Moskva: Fizmatgiz, 1959. – 232 s.
[5] Elubaev S.E., Dilman T.B. Giperbolalykh zhane parabolalykh tendeuler ushin kejbir keri esepter. 3-basylymy. Almaty: Evero, 2016. – 184 b.
[6] Dilman T.B. Edinstvennost I ustoichivost nekotorykh zadach integralnoj geometii. – Saarbrucken: LAP LAMBERT Academic Publishing, 2016. – 58 p.